1. Let $V = M_{2\times 2}(\mathbb{R})$ and $T \in \mathcal{L}(V)$ is defined by

$$T(A) = A + A^T$$

Determine whether T is diagonalizable. Please explain your answers with details.

let $\beta = \{E_{11}\ E_{12}\ E_{21}\ E_{22}\}$ be the standard ordered basis for V.

$$\begin{cases} T(E_{11}) = 2E_{11} \\ T(E_{12}) = E_{12} + E_{21} \\ T(E_{21}) = E_{12} + E_{21} \\ T(E_{22}) = 2E_{22} \end{cases} \qquad [T]_\beta = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

$$\begin{aligned} f_T(t) &= \det([T]_\beta - tI_4) \\ &= (2-t)^2 \cdot ((1-t)^2 - 1) \\ &= (2-t)^2\ t\ (t-2) \\ &= (t-2)^3\ t \end{aligned}$$

The eigenvalues are 0 and 2

- $1 \leq \gamma_T(0) \leq \mu_T(0) = 1$ Thus $\mu_T(0) = \gamma_T(0) = 1$
- $\mu_T(2) = 3$

$$[T]_\beta - 2I_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_T(2) = \dim(N([T]_\beta - 2I_4)) = 4 - \operatorname{rank}([T]_\beta - 2I_4) = 3 = \mu_T(2)$$

Therefore T is diagonalizable

2. Let $V = P_2(\mathbb{R})$ and $T \in \mathcal{L}(V)$ is defined by

$$T(a + bx + cx^2) = (-a - 2b + c) - (\frac{1}{2}c)x + (2b + 2c)x^2.$$

(a) Find a polynomial $g(t)$ of degree at most 2 such that $T^3 = g(T)$. (**Hint:** Cayley-Hamilton Theorem.)

(b) Let $\mathbf{v} = -x + 2x^2 \in V$ and W be the T-cyclic subspace of V generated by $\mathbf{v}$. Show that $T^2(\mathbf{v}) = a_0\mathbf{v} + a_1T(\mathbf{v})$ for some $a_0, a_1 \in \mathbb{R}$. What's $\dim(W)$? Find the characteristic polynomial of $T|_W$, the restriction of T to W.

(a) $A = [T]_\beta = \begin{pmatrix} -1 & -2 & 1 \\ 0 & 0 & -\frac{1}{2} \\ 0 & 2 & 2 \end{pmatrix}$ $\quad f_T(t) = \det(A - tI_3) = \det \begin{pmatrix} -1-t & -2 & -1 \\ 0 & -t & -\frac{1}{2} \\ 0 & 2 & 2-t \end{pmatrix}$

$= (-1-t) \cdot \left(-t(2-t) - 2 \cdot \frac{-1}{2}\right)$

$= -(t+1) \cdot (t^2 - 2t + 1)$

$= -(t+1) \cdot (t-1)^2 \quad = -t^3 + t^2 + t - 1$

by Cayley-Hamilton Thm $\quad f_T(T) = T_0$

let $g(t) = t^2 + t - 1$ then $T^3 = g(T)$.

(b) $v = 0 - x + 2x^2$

$T(v) = 4 - x + 2x^2$

$T^2(v) = 0 - x + 2x^2 \qquad$ Then $T^2(v) = 1 \cdot v + 0 \cdot T(v)$

Since $\{v, T(v)\}$ linearly independent but $\{v, T(v), T^2(v)\}$ linearly dependent

$\dim(W) = \dim\left(\text{span}(\{v, T(v), T^2(v), \ldots\})\right) = 2$

Since $T^2(v) - 0 \cdot T(v) - 1 \cdot v = 0$

$f_{T_W}(t) = (-1)^2 \cdot (t^2 - 0t - 1) = t^2 - 1$

3. Let V be a vector space over $\mathbb{C}$ with an ordered basis $\beta = \{\mathbf{v}_0, \mathbf{v}_1, ..., \mathbf{v}_{n-1}\}$. Define a linear operator $T : V \to V$ by:

$$T(\mathbf{v}_0) = \mathbf{v}_0 - \mathbf{v}_{n-1} \text{ and } T(\mathbf{v}_k) = \mathbf{v}_k - \mathbf{v}_{k-1} \text{ for } 1 \le k \le n-1.$$

Let $\omega_k = e^{i\frac{2\pi k}{n}} = \cos(\frac{2\pi k}{n}) + i\sin(\frac{2\pi k}{n})$ for any integer k (where $i = \sqrt{-1}$).

(a) Show that $\mathbf{u}_k = \sum_{j=0}^{n-1} \omega_k^j \mathbf{v}_j$ is an eigenvector of T for any integer k and show that T is diagonalizable. (**Hint:** You may use the fact that $\omega_k^j = e^{i\frac{2\pi kj}{n}}$ and $\omega_k^n = 1$.)

(b) Now, consider the linear operator $U : V \to V$ defined by: $U(\mathbf{v}_0) = \mathbf{v}_1 - 2\mathbf{v}_0 + \mathbf{v}_{n-1}$, $U(\mathbf{v}_k) = \mathbf{v}_{k-1} - 2\mathbf{v}_k + \mathbf{v}_{k+1}$ for $1 \le k \le n-2$ and $U(\mathbf{v}_{n-1}) = \mathbf{v}_{n-2} - 2\mathbf{v}_{n-1} + \mathbf{v}_0$. Using (a), determine if there exists an ordered basis γ for V such that $[U]_\gamma$ is a **real** diagonal matrix. Please explain you answer with details.

(a) $T(u_k) = T(\sum_{j=0}^{n-1} w_k^j v_j) = \sum_{j=0}^{n-1} w_k^j T(v_j)$

$= w_k^0 (v_0 - v_{n-1}) + w_k^1 (v_1 - v_0) + \cdots + w_k^{n-1} (v_{n-1} - v_{n-2})$

$= (w_k^0 - w_k^1) v_0 + (w_k^1 - w_k^2) v_1 + \cdots$

$+ (w_k^{n-2} - w_k^{n-1}) v_{n-2} + (w_k^{n-1} - \underbrace{w_k^0}_{= w_k^n}) \cdot v_{n-1}$

$= (1 - w_k) \cdot (v_0 + w_k^1 v_1 + \cdots + w_k^{n-2} \cdot v_{n-2} + w_k^{n-1} \cdot v_{n-1})$

$= (1 - w_k) u_k. \quad \forall k \in \mathbb{Z}.$

Since eig. values $\{1-w_0, 1-w_1, \cdots, 1-w_{n-1}\}$ are distinct.

That is T has n distinct eigen values, T is diagonalizable

(b) $U(u_k) = U(\sum_{j=0}^{n-1} w_k^j v_j) = \sum_{j=0}^{n-1} w_k^j \cdot U(v_j)$

$= w_k^0 (v_{n-1} - 2v_0 + v_1) + w_k^1 (v_0 - 2v_1 + v_2) + \cdots$

$+ w_k^{n-1} (v_{n-2} - 2v_{n-1} + v_0)$

$= (-2w_k^0 + w_k^1 + w_k^{n-1}) v_0 + (-2w_k^1 + w_k^0 + w_k^2) v_1 + \cdots$

$+ (-2w_k^{n-1} + w_k^{n-2} + w_k^0) v_{n-1}$

$= (-2 + w_k + w_k^{-1}) \cdot (v_0 + w_k^1 v_1 + \cdots + w_k^{n-1} \cdot v_{n-1})$

$= \underbrace{(-2 + w_k + w_k^{-1})}_{\in \mathbb{R}} \cdot u_k$

Let $\gamma = \{u_0 \cdots u_{n-1}\}$.

By (a). γ are eigen.vectors of T corresponding to distinct eigen values

Then γ is L.I. and a basis for V.

$[U]_\gamma = \text{diag}((-2+w_0+w_0^{-1}, -2+w_1+w_1^{-1}, \cdots, -2+w_{n-1}+w_{n-1}^{-1}))$ is real diag matrix.

4. Let F be a field and $V = F^n$ be a vector space over F. Let $\Phi : V^* \to F^n$ be defined by $\Phi(f) = (c_1, \ldots, c_n)$, where $f(\vec{x}) = f(x_1, \ldots, x_n) = \sum_{i=1}^n c_i x_i$. Let $W = \{\vec{x} = (x_1, \ldots, x_n) \in V : \sum_{i=1}^n x_i = 0\}$ be a subspace of V.

(a) Let $\vec{v}_0 = (1, 1, \ldots, 1) \in V$. Let $\eta : W^* \to V^*$ be a linear map between W^* and V^* such that $\eta(g)(\vec{x}) = \begin{cases} g(\vec{x}) & \vec{x} \in W \\ 0 & \vec{x} \in \text{span}(\{\vec{v}_0\}) \end{cases}$. Show that η is well-defined. (That is, for each $g \in W^*$, $\eta(g) \in V^*$ is uniquely determined.)

(b) Show that $R(\Phi \circ \eta) = \{(c_1, \ldots, c_n) \in F^n : \sum_{i=1}^n c_i = 0\}$.

(a) Claim : $V = W \oplus \text{span}(\{\vec{v}_0\})$

Pf. Let $\beta' = \{(1,-1,0,\cdots,0), (1,0,-1,0,\cdots,0) \cdots (1,0,\cdots,0,-1)\}$ be a basis for W.

Since $\beta = \beta' \cup \{\vec{v}_0\}$ is L.I. and $\dim(V) = n = |\beta|$.

β is a basis for V.

$V = \text{span}(\beta) = \text{span}(\beta') \oplus \text{span}(\{\vec{v}_0\}) = W \oplus \text{span}(\{\vec{v}_0\})$

Then $\forall \vec{x} \in V$. $\exists! \vec{w} \in W$. and $! a \in F$ s.t

$\vec{x} = \vec{w} + a \cdot \vec{v}_0$

$\eta(g)(\vec{x}) = \eta(g)(\vec{w} + a\vec{v}_0) = \eta(g)(\vec{w})$

is uniquely determined.

Thus $\eta(g)$ is well-defined.

(b)

- Φ is isomorphism

$\Phi : V^* \to F^n$

$f \mapsto (c_1 \cdots c_n)$ where $f(x_1 \cdots x_n) = \sum_1^n c_i x_i$

Φ is well-defined, linear, 1-1, onto.

- η is 1-1.

$\forall g \in N(\eta)$. $\eta(g) = 0$ i.e. $\eta(g)(\vec{x}) = 0 \quad \forall \vec{x} \in V$

Then $g(\vec{w}) = \eta(g)(\vec{w}) = 0 \quad \forall \vec{w} \in W \subset V$

i.e. $g = 0$.

$$W^* \xrightarrow{\eta} V^* \xrightarrow{\Phi} F^n.$$

① $\forall g \in W^*$. $\quad \eta(g)(\vec{x}) = \begin{cases} g(\vec{x}) & \vec{x} \in W \\ 0 & \vec{x} \in \mathrm{span}(\{\vec{v}_0\}) \end{cases}$

$\eta(g) \in V^*$. So let $\eta(g)(\vec{x}) = \sum_1^n c_i \cdot x_i$

Since $\eta(g)(\vec{v}_0) = 0$ we have $\sum_1^n c_i = 0$

i.e. $\Phi(\eta(g)) = (c_1 \cdots c_n) \in \{(c_1 \cdots c_n) : \sum_1^n c_i = 0\}$

$R(\Phi \circ \eta) \subset \{(c_1 \cdots c_n) : \sum c_i = 0\}$

② Since η is 1-1, Φ is isomorphism

$$\begin{cases} \dim(R(\Phi \circ \eta)) = \dim(W^*) = \dim(W) = n-1 \\ \dim(\{(c_1 \cdots c_n) : \sum_1^n c_i = 0\}) = n-1 \end{cases}$$

By ① and ②. we know $R(\Phi \circ \eta) = \{(c_1 \cdots c_n) : \sum c_i = 0\}$

5. Let V be a finite-dimensional vector space over $\mathbb{R}$ with an ordered basis $\beta = \{\mathbf{v}_i\}_{i=1}^n$. Consider a linear transformation $\Phi : V \otimes V \to \mathcal{L}(V^*, V)$, which is defined by:

$$\Phi(\mathbf{v}_i \otimes \mathbf{v}_j)(f) = f(\mathbf{v}_i)\mathbf{v}_j \text{ for all } f \in V^*.$$

(a) Prove that Φ is an isomorphism and $[\Phi(\mathbf{w}_1 \otimes \mathbf{w}_2)]_{\beta^*}^{\beta} = ([\mathbf{w}_2]_\beta)([\mathbf{w}_1]_\beta)^T$.

(b) Let $\{\mathbf{w}_1, ..., \mathbf{w}_m\}$ be a linearly independent subset of V and A be a $m \times m$ real matrix. Consider $G = \sum_{i=1}^m \sum_{j=1}^m A_{ij}\mathbf{w}_i \otimes \mathbf{w}_j$, where A_{ij} is the i-th row j-th column entry of A. Find the rank of $\Phi(G)$.

(a)

let $[w_1]_\beta = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ $\quad [w_2]_\beta = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$

$$\Phi(w_1 \otimes w_2) = \Phi\left(\left(\sum_{i=1}^n a_i v_i\right) \otimes \left(\sum_{j=1}^n b_j v_j\right)\right) = \sum_{i,j=1}^n a_i b_j \, \Phi(v_i \otimes v_j)$$

Let $\beta^* = \{f_1 \cdots f_n\}$ be dual basis of β.

Then $\quad \Phi(w_1 \otimes w_2)(f_k) = \sum_{i,j=1}^n a_i b_j \, \Phi(v_i \otimes v_j)(f_k) = \sum_{i,j=1} a_i b_j \, f_k(v_i) \cdot v_j = a_k \sum_{j=1}^n b_j v_j$

$$[\Phi(w_1 \otimes w_2)]_{\beta^*}^{\beta} = \begin{pmatrix} a_1 b_1 & \cdots & a_k b_1 & \cdots & a_n b_1 \\ \vdots & & \vdots & & \vdots \\ a_1 b_n & \cdots & a_k b_n & \cdots & a_n b_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \cdot (a_1 \cdots a_n) = [w_2]_\beta \cdot [w_1]_\beta^T$$

(b)

$A \in M_{m \times m}(\mathbb{R})$

$\{w_1 \cdots w_m\}$ L. I. $\quad$ extend to $\gamma = \{w_1 \cdots w_m \cdots w_n\}$ basis for V.

dual basis $\gamma^* = \{g_1 \cdots g_m \cdots g_n\}$ for V^*

$\Phi(G) : V^* \to V$

$$\begin{aligned} \Phi(G)(f) &= \Phi\left(\sum_{i=1}^m \sum_{j=1}^m A_{ij} w_i \otimes w_j\right)(f) \\ &= \sum_{i,j=1}^m A_{ij} \, \Phi(w_i \otimes w_j)(f) \\ &= \sum_{i,j}^m A_{ij} \, f(w_i) \cdot w_j \end{aligned}$$

$$\Phi(G)(g_k) = \sum_{i=1}^m \sum_{j=1}^m A_{ij} \, g_k(w_i) \cdot w_j = \begin{cases} \sum_{j=1}^m A_{kj} w_j & 1 \le k \le m \\ 0 & m+1 \le k \le n \end{cases}$$

$$[\Phi(G)]_{\gamma^*}^{\gamma} = \begin{bmatrix} A^T & O_{m \times (n-m)} \\ O_{(n-m) \times m} & O_{(n-m) \times (n-m)} \end{bmatrix}$$

$$\text{rank}(\Phi(G)) = \text{rank}\begin{pmatrix} A^T & 0 \\ 0 & 0 \end{pmatrix} = \text{rank}(A)$$